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DIFFERENT REALIZATIONS OF THE STASHEFF POLYTOPE

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ABSTRACT

For the importantance of the Stasheff polytope in different area of mathematics and the relationship between them we studied the Stasheff polytope and the permutohedron in view of combinatorial and algebra using the method that computer the coordinate of their vertices. The relationship between the associahedron and the permutohedron are also explained with the aid of graph theory.

KEYWORDS: Stasheff Polytope, Permutohedron

INTRODUCTION

The Stasheff polytope K^n (associahedron), appeared in the sixties of the work Jim of Stasheff on the recognition of loop space, [5].

The associahedron K^n is the convex hull of points for the vertices of the associahedron which represented by planar binary tree Y_n with (n+1) leaves. Let S_n be the symmytric group and M (σ) = $(\sigma(1),\dots,\sigma(n)) \in R^n$. The convex hull of the points M (σ), for $\sigma \in S_n$ represent (n-1) – dimensional polytope denoted by P^{n-1} and called it the permutohedron.

In the Euclidian space R^n the coordinates of a point are denoted by $x_1, \dots, x_n$, and the hyperplane H is represented by the equation $\sum_{i=1}^{n} x_i = \frac{1}{2}\, n\,(n+1)$, Loday in [5] showed that $P^{n-1} \subseteq K^{n-1}$, (by truncation) such that the permutahedron can be obtained by truncating the standard simplex along some hyperplanes, one each cell of Δ^{n-1} (except the big cell), where 2^{n-1} points are common vertices of K^{n-1} and P^{n-1}, The main idea of our work is to prove Lodays result using another method.

PRELIMIARIES

Definition (1), [1]

Let Ax ≤ b, where A∈ $R^{m\,x\,d}$ is a given real matrix, and b∈ R^m is a known vector. A set P ={ x ∈ R^d : Ax ≤ b } is said to be a polyhedron. A polyhedron P is bounded if there exists M ∈ R^+, such that $\|x\| \leq$ M for every x ∈ P. Every bounded polyhedron is said to be a polytope, as seen in figure 1.

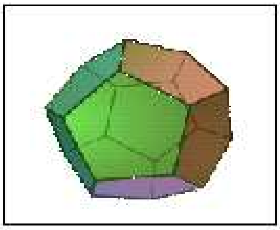


Figure 1: A Polytope

Definition (2), [3]

A graph G consists of two finite sets V and E each element of V is called vertex and each element of E is called edge. An edge of G represented as unordered pairs of vertices. A graph G is connected if every pair of vertices can be joined by a path, as seen is figure 2.

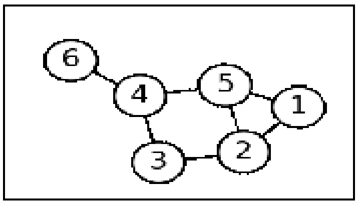


Figure 2: Graph

Definition (3), [3]

A connected graph that contains no cycles is called a tree, as seen in figure 3.

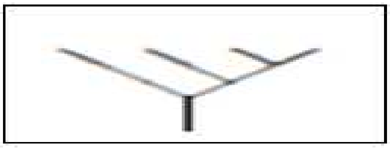


Figure 3: Tree

Note (1), [6]

Let Y_n be the set of planar binary trees with n internal vertices, observe that there is a bijection between the set of trees and the set of parenthesizing of a word with n+1 letters as shown in the following figure.

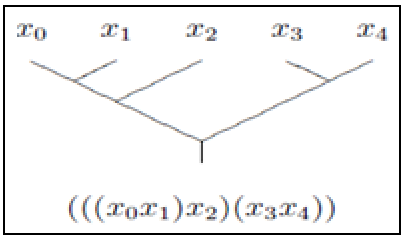


Figure 4: Parenthesizings

Definition (4), [4]

Let S_n be the symmetric group on the set {1,…, n}. An (k ,n-k) –shuffle is a permutation $\sigma \in S_n$ where σ = (σ (1),…, σ (k) $|$ σ (k+1),…, $\sigma(n)$) ∈ S_n such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(\mathrm{k}+1) < \sigma(\mathrm{k}+2) < \sigma(\mathrm{n})$ where ($\sigma(1), \dots, \sigma(n)$) is the image of {1,...,n} under the permutation σ.

Definition (5), [1]

A set of points X = $\{x_i\}$ for i = {1,...,n} which satisfies $\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$, , is said to be a hyper plane.

PLANAR BINARY TREES

Now, some definitions and properties for the planar binary trees are given below:

Definition (6), [4]

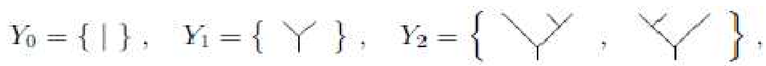
A planar tree is an oriented graph drawn on a plane, with only one root. Its binary if any root is trivalent (one root and two leaves).

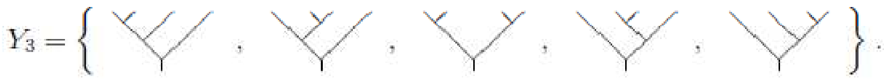
Note (2), [4]

Let Y_n be the set of all planar binary trees with (n+1) leaves, with n interior vertices The number of element in Y_n is the Catalan number C_n = (2n)! / (n! (n+1)!).

Example (1), [4]

For the planar binary trees,





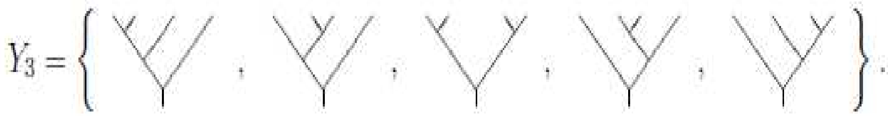
The Catalan number of each them is $c_0 = 1, c_1 = 1, c_2 = 2, and\ c_3 = 5$.

Definition (7), [2]

A leveled tree (or ordered tree) is a planar binary tree such that each node must be on distinct horizontal level.

Example (2), [4]

Let Y_3 be the set of all binary trees as given below:



Then the leveled and un leveled tree are shown by figure 5

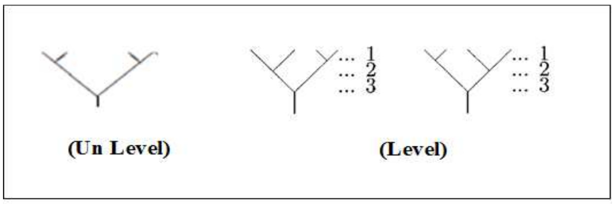


Figure 5: Level and un Leveled

Note (3)

Let T_n be the set of all leveled trees with n+1 leaves and n nodes, then the number of trees in T_n is n!, [6]; as given in example (3.3).

Example (3), [2]

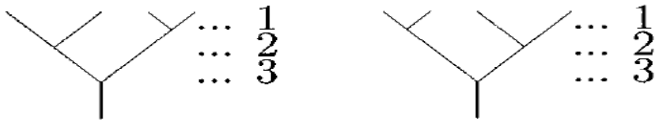
For T_3 the number of leveled trees with 4 leaves is 6 which are

Note (4)

For any leveled tree one can associated a permutation by labeling the vertices from 1,2,3,…,n according to the level. Then the permutation is the leveled number from left to right.

Example (4), [4]

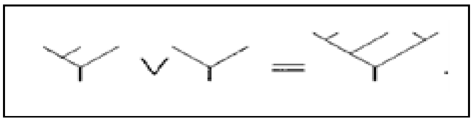
For the leveled trees



The permutations are (231) and (132) respectively.

Definition (8), [4]

The grafting (p+q+1)-tree is the joining for the roots of p- tree T_1 and a q- tree T_2 that create a new root. For instance,



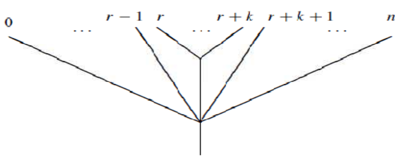
Definition (9), [5]

Let $T_n{}^*$ be the set of planar trees with n+1 leaves n ≥ 0 (and one root) such that the valence of each internal vertex is at least 2, and $T_n{}^*$ = $T_{n,1}{}^*$ ∪… ∪ $T_{n,n}{}^*$ where $T_{n,k}{}^*$ made of the planar trees which have n – k + 1 internal vertices.

Example (5), [5]

Definition (10), [5]

A tree t is said to be a refinement of the tree t' if t' can be obtained from t by contracting to a point for some of the internal edges, any tree in $T_{n,n-1}{}^*$ is of the form



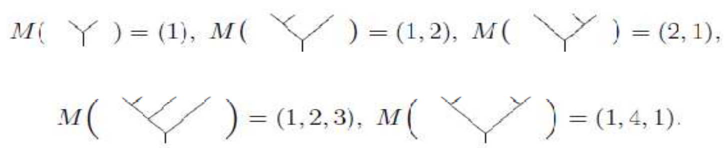
THE ASSOCIHEDRON K^n

In this section we the represent the trees as a point in R^n with integral coordinate using the following steps:

- Numbered the leaves of t from left to right by 0, 1, 2,…,n.
- Numbered the internal vertices from 1 to n. 3. Represented the number of leaves on the left side by a_i of the vertex i, and represented the number of leaves on the right side byb_i, so M(t) = ($a_1\ b_1, \ldots, a_i\ b_i\ , \ldots, a_n\ b_n$) ∈ R^n .

The following example represents how to convert tree to a point in R^n.

Example (6), [6]



Note (6), [7]

The associahedron of dimension n is the convex hull of the point M(t) for all planar binary trees with (n+1) leave. The sum of the coordinate x_i for M(t) is $\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$, So the associahedron lies in the hyper plane given by this equation, that is for any tree t ∈ Y_n the coordinates of the point M(t)=($x_1, \ldots, x_n$)∈ R^n satisfy the relation $\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$ Hence M (t) ∈H.

As an example is given by figure (6), [8]

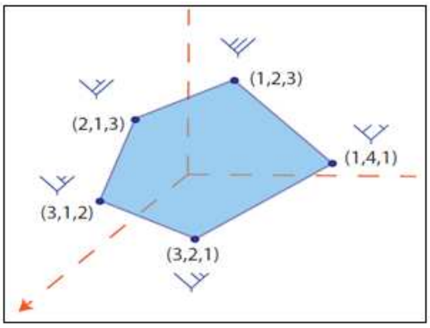


Figure 6: The Associahedron

THE PERMUTOHEDRON P^{n-1}

Another example of a polytope related to a combinatorial structure is the permutohedron. An element σ in the symmetric group S_n, a associated to the point M(σ) which is equal to (σ (1),…, σ (n)) ∈ R^n.

The permutohedron P^{n-1} is a convex hull of all points M (σ) for all $\sigma \in S_n$. The sum of coordinates is

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$$

So the permutohedron lies in the hyperplane given by the above equation [7]. As an example is given by figure (7)

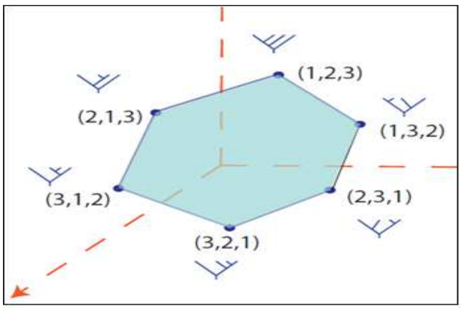


Figure 7: The Permutohedron

RELATING THE PERMUTOHEDRON AND ASSOCIAHEDRON

The main purpose of this chapter is to prove Lodays result using another way.

We need the following lemma to prove our theorem.

Lemma (1), [4]

There is a subjective map from S_n to Y_n.

Note (7)

The subjective map given in lemma 1 convert every permutation to a binary tree using the following,

The image of {1, … , n } under the permutation σ is a sequence of positive integers [$\sigma_1 , \dots , \sigma_n$]. Hence

- Replace the largest integer in the interval $\sigma_1 , \dots , \sigma_n$ say σ_p , by the length of the interval (which is n here).
- Repeat the modification for the intervals $\sigma_1 , \dots , \sigma_{p-1}$ *and* $\sigma_{p+1} , \dots , \sigma_n$, until each integer has been modified, This gives the name of tree.
- Using the grafting procedure on the name of tree to get the tree

Now, we prove our result.

Theorem (1)

Any Permutohedron P^{n-1} is contained in the associahedron K^{n-1} for all n ≥ 1.

Proof

Firstly, we convert the permutohedron P^{n-1} to the set of binary trees Y_n, by lemma (1). Let Q_n = $\{0,1\}^{n-1}$ and ∅ : $Y_n \rightarrow Q_n$, defined by ∅ (t) = (ϵ_1,…,ϵ_{n-1}) $\forall\ t \in Y_n$, where ϵ_i =0 (resp. ϵ_i = 1) if the i th leaf of t is pointing to the left (resp. to the right), we code the vertices of the cube by the elements of Q_n. The convex polytope C^{n-1} which is defined in H by the equations P_ω(M) ≥ 0 for $\omega = (1\ 2\ \dots\ i\ \vert$ i+1…..n) and $\omega = (\ i+1 \dots n|1\ 2\ ..\ i), i = 0\ , \dots , n-1,$ then the facetes of C^{n-1} are in the hyperplanes $H_{1,2\dots n}$, $H_{12,\dots n}$,…, $H_{1\dots n,n}$ and $H_{2\dots n,1}$, $H_{3\dots n,12}$,…, $H_{n,1\dots n-1.}$ The coordinate of the vertex M(ϵ)= (x_1, ..., x_n) for $\epsilon = (\epsilon_1 , \dots , \epsilon_{n-1}) \in Q_n$ are x_i = i - ϵ_{i-1}(n-i+1) (i-1) + ϵ_i (n- i) (i), for i = 1 , … , n, [7].

Now, we truncate every point in the cube which is give two levels trees and every point in the cube resulting from grafting two trees and refinement in the associahedron using the hyperplanes and its dual to get the permutahedron

The following example explain theorem (7.1).

Example (7)

Let S_3 be the symmetric group. For any σ∈ S_3, we get

S_3= {(123), (132), (321), (213), (312), (231) } and

(1|2|3) = (123), (2|1|3) = (213), (2|3|1)=(231), (3|2|1)=(231), (3|1|2)=(312),

(1|3|2) = (132)

$\psi : S_3 \rightarrow Y_3$, By lemma 1, we get

For σ = (123), then ψ (123) = (123) =  ,

Similarly, we set

σ = (321), then ψ (321) = (321) =  ,

σ = (213), then ψ (213) =  ,

σ = (312), then ψ (312) = (312) =  ,

σ = (231), then ψ (231) = (131) =  ,

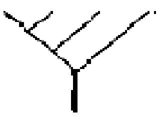
σ = (132), then ψ (132) = (131) =  ,

The following planar binary tree is obtained

Now, Q_3 = { 0, 1}^{2} = { (0,0), (0,1), (1,0), (1,1) }.

M (ϵ) = {x_1, x_2 , x_3 } , ϵ = (ϵ_1,ϵ_2),

x_i = i - ϵ_{i-1}(n-i+1) (i-1) + ϵ_i (n- i) (i), for i = 1 , … , n.

 , we get ϵ = (0,1)

x_1 = 1- ϵ_0 (3-1+1)(1-1) + ϵ_1 (3-1)(1) = 1+0 =1,

x_2 = 2- ϵ_1 (3-2+1)(2-1) + ϵ_2 (3-2)(2) = 2+2 =4, and

x_3 = 3- ϵ_2 (3-3+1)(3-1) + ϵ_3 (3-3)(3) = 3-2 =1, then

M (ϵ) = (1, 4, 1) , … etc

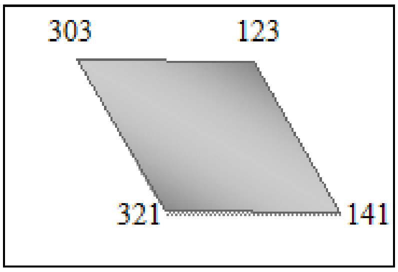


Figure 8: The Cube

Now,

- Apply definition (2.5) on the set of coordinate to get the hyperplane and its dual.
- Truncate every point in the cube which is given by two levels trees.
- Truncate every point in the cube resulting from grafting and a refinement in the associahedron to get the permutahedron.

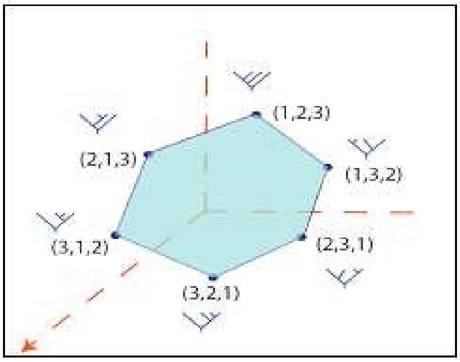


Figure 9: The Permotuhedron

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